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Integrable Poisson algebras and two-dimensional manifolds

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Abstract. The relations between integrable Poisson algebras with three generators and two-dimensional symplectic manifolds are investigated. It is shown that for a given integrable Poisson algebra $\mathcal{A}$ there exists a two-dimensional symplectic manifold $M \subset \mathbb{R}^3$ such that the Poisson algebra generated by the coordinates of M coincides with the algebra $\mathcal{A}$. *Vice versa* the coordinates of a given smooth two-dimensional symplectic manifold M embedded in $\mathbb{R}^3$ generate an integrable Poisson algebra. Moreover, smooth Poisson algebraic maps between two integrable Poisson algebras are governed by equations involving the symplectic manifolds corresponding to these algebras.

Poisson algebras have been discussed widely in Hamiltonian mechanics and in the quantization of classical systems, such as canonical quantization and Moyal product quantization, see for example [1–5]. In this paper we investigate the relations between symplectic manifolds and Poisson algebras. We find that there exist general relations between integrable Poisson algebras with three generators and two-dimensional symplectic manifolds.

We first recall some basic knowledge of symplectic geometry. A symplectic manifold (M, ω) is an even-dimensional manifold M equipped with a symplectic 2-form ω, see for example [6–9]. Let d denote the exterior derivative on M. By definition a symplectic form ω on M is closed, $d\omega = 0$, and non-degenerate, $X \rfloor \omega = 0 \Rightarrow X = 0$, where X is a (smooth) vector on M and $\rfloor$ denotes the left inner product defined by $(X \rfloor \omega)(Y) = \omega(X, Y)$ for any two vectors X and Y on M. The non-degeneracy means that for every tangent space $T_x M$, $x \in M$ and with $X \in T_x M$, the relation $\omega_x(X, Y) = 0$ for all $Y \in T_x M$ implies $X = 0$.

Infinitesimal symplectic diffeomorphisms are given by vectors. A vector X on M corresponds to an infinitesimal canonical transformation if and only if the Lie derivative of ω with respect to X vanishes,

$$\mathcal{L}_X \omega = X \rfloor d\omega + d(X \rfloor \omega) = 0. \tag{1}$$

A vector X satisfying (1) is said to be a Hamiltonian vector field.

Since ω is closed, it follows from (1) that a vector X is a Hamiltonian vector field if and only if $X \rfloor \omega$ is closed. Since ω is non-degenerate, this gives rise to an isomorphism between vector fields X and 1-forms on M given by $X \to X \rfloor \omega$. Let $\mathcal{F}(M)$ denote the

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real-valued smooth functions on M. For an $f \in \mathcal{F}(M)$, there exists a Hamiltonian vector field X_f (unique up to a sign on the right-hand side of the following equation) satisfying

$$X_f \rfloor \omega = -\mathrm{d}f. \tag{2}$$

X_f is called the Hamiltonian vector field associated with f.

Let $f, g \in \mathcal{F}(M)$. The Lie bracket $[X_f, X_g]$ is the Hamiltonian vector field of $\omega(X_f, X_g)$, in the sense that

$$\begin{aligned}[X_f, X_g] \rfloor \omega &= \mathcal{L}_{X_f}(X_g \rfloor \omega) - X_g \rfloor (\mathcal{L}_{X_f}\omega) \\ &= X_f \rfloor \mathrm{d}(X_g \rfloor \omega) + \mathrm{d}(X_f \rfloor (X_g \rfloor \omega)) - X_g \rfloor \mathrm{d}(X_f \rfloor \omega) \\ &= -\mathrm{d}(\omega(X_f, X_g))\end{aligned}$$

where the Cartan formula for the Lie derivative $\mathcal{L}_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X$ of a vector X has been used. The function $-\omega(X_f, X_g)$ is called the Poisson bracket of f and g and denoted by $[f, g]_{\mathrm{PB}}$,

$$[f, g]_{\mathrm{PB}} = -\omega(X_f, X_g) = -X_f g. \tag{3}$$

Since ω is closed, the so-defined Poisson bracket satisfies the Jacobi identity

$$[f, [g, h]_{\mathrm{PB}}]_{\mathrm{PB}} + [g, [h, f]_{\mathrm{PB}}]_{\mathrm{PB}} + [h, [f, g]_{\mathrm{PB}}]_{\mathrm{PB}} = 0.$$

Therefore, under the Poisson bracket operation the space $C^\infty(M)$ of all smooth functions on (M, ω) is a Lie algebra, called the Poisson algebra of (M, ω).

In general, one calls a Poisson algebra any associative, commutative algebra $\mathcal{A}$ over $\mathbb{R}$ with unit, equipped with a bilinear map $[,]_{\mathrm{PB}}$, called a Poisson bracket satisfying:

(1) *antisymmetry*

$$[f, g]_{\mathrm{PB}} = -[g, f]_{\mathrm{PB}}$$

(2) *derivation property*

$$[fg, h]_{\mathrm{PB}} = f[g, h]_{\mathrm{PB}} + g[f, h]_{\mathrm{PB}}$$

(3) *Jacobi identity*

$$[f, [g, h]_{\mathrm{PB}}]_{\mathrm{PB}} + [g, [h, f]_{\mathrm{PB}}]_{\mathrm{PB}} + [h, [f, g]_{\mathrm{PB}}]_{\mathrm{PB}} = 0$$

for any $f, g, h \in \mathcal{A}$.

Now let $\mathcal{A}$ be a Poisson algebra with three generators $(x_1, x_2, x_3) = x$ and a Poisson bracket of the form

$$[x_i, x_j]_{\mathrm{PB}} = \sum_{k=1}^{3} \epsilon_{ijk} f_k \tag{4}$$

where ϵ_{ijk} is the completely antisymmetric tensor and f_i, $i = 1, 2, 3$, are smooth real-valued functions of x, restricted to satisfy the Jacobi identity:

$$\begin{aligned}&[x_1, [x_2, x_3]_{\mathrm{PB}}]_{\mathrm{PB}} + [x_2, [x_3, x_1]_{\mathrm{PB}}]_{\mathrm{PB}} + [x_3, [x_1, x_2]_{\mathrm{PB}}]_{\mathrm{PB}} \\ &\quad = \frac{\partial f_1}{\partial x_2} f_3 - \frac{\partial f_1}{\partial x_3} f_2 + \frac{\partial f_2}{\partial x_3} f_1 - \frac{\partial f_2}{\partial x_1} f_3 + \frac{\partial f_3}{\partial x_1} f_2 - \frac{\partial f_3}{\partial x_2} f_1 = 0.\end{aligned}$$

We say that the Poisson algebra (4) is integrable if f_i satisfies

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad i, j = 1, 2, 3 \tag{5}$$

and at least one of f_i, $i = 1, 2, 3$, is non-zero. Obviously the integrability condition (5) is a sufficient condition for the Poisson algebra (4) to satisfy the Jacobi identity.

Let $\mathcal{F}$ be the space of smooth real-valued functions of x, $x \in \mathbb{R}^3$. We consider the realization of the Poisson algebra $\mathcal{A}$ in $\mathbb{R}^3$ and will not distinguish between the symbols x_i of the coordinates of $\mathbb{R}^3$ and the generators of $\mathcal{A}$. In the following M will always denote a smooth two-dimensional manifold smoothly embedded in $\mathbb{R}^3$.

Theorem 1. For a given integrable Poisson algebra $\mathcal{A}$ there exists a two-dimensional symplectic manifold M described by an equation of the form $F(x) = c$, $x \in \mathbb{R}^3$, with $F \in \mathcal{F}$ and c an arbitrary real number, such that the Poisson algebra generated by the coordinate functions x_1, x_2, x_3 of $\mathbb{R}^3$ restricted to M coincides with the algebra $\mathcal{A}$.

Proof. A general integrable Poisson algebra is of the form (4),

$$[x_i, x_j]_{\mathrm{PB}} = \sum_{k=1}^{3} \epsilon_{ijk} f_k$$

where f_i, $i = 1, 2, 3$, satisfy the integrability condition (5). What we have to show is that this Poisson algebra can be described by the symplectic geometry on a suitable two-dimensional symplectic manifold (M, ω), in the sense that the above Poisson bracket can be described by the formula (3), i.e. the Poisson bracket $[x_i, x_j]_{\mathrm{PB}}$ is given by the Hamiltonian vector field X_{x_i} associated with x_i such that

$$[x_i, x_j]_{\mathrm{PB}} = -X_{x_i} x_j = \sum_{k=1}^{3} \epsilon_{ijk} f_k \tag{6}$$

with x_i the coordinates of the two-dimensional manifold M in $\mathbb{R}^3$.

Let $X'_{x_i} \in \mathbb{R}^3$ be given by

$$X'_{x_i} \equiv \sum_{j,k=1}^{3} \epsilon_{ijk} f_j \frac{\partial}{\partial x_k}. \tag{7}$$

Then X'_{x_i} satisfies

$$-X'_{x_i} x_j = \sum_{k=1}^{3} \epsilon_{ijk} f_k.$$

A general 2-form on $\mathbb{R}^3$ can be written as

$$\omega' = -\frac{1}{2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} h_i \, \mathrm{d}x_j \wedge \mathrm{d}x_k \tag{8}$$

where $h_i \in \mathcal{F}$, $i = 1, 2, 3$. We have to prove that x can be restricted to a suitable two-dimensional manifold $M \subset \mathbb{R}^3$ in such a way that X'_{x_i} coincides with the Hamiltonian vector field X_{x_i} and ω' is the corresponding symplectic form on M.

A 2-form on a two-dimensional manifold is always closed. What we should then check is that there exists $M \subset \mathbb{R}^3$ such that for x restricted to M the formula (2) holds for $f = x_i$, i.e.

$$X'_{x_i} \rfloor \omega' = -\mathrm{d}x_i \qquad x_i \in M, i = 1, 2, 3. \tag{9}$$

Substituting formulae (8) and (7) into (9) we get

$$X'_{x_i} \rfloor \omega' = -\sum_{j,k=1}^{3} \epsilon_{ijk} f_j \frac{\partial}{\partial x_k} \rfloor \frac{1}{2} \sum_{l,m,n=1}^{3} \epsilon_{lmn} h_l \, \mathrm{d}x_m \wedge \mathrm{d}x_n$$

$$= -\frac{1}{2}\sum_{lmnjk}^{3}\epsilon_{ijk}\epsilon_{lmn} f_j h_l (\delta_{km}\,\mathrm{d}x_n - \delta_{kn}\,\mathrm{d}x_m)$$

$$= -\sum_{lnjk}^{3}\epsilon_{ijk}\epsilon_{lkn} f_j h_l\,\mathrm{d}x_n = -\mathrm{d}x_i.$$

That is,

$$\begin{aligned}
&(1 - f_2h_2 - f_3h_3)\,\mathrm{d}x_1 + f_2h_1\,\mathrm{d}x_2 + f_3h_1\,\mathrm{d}x_3 = 0\\
&(1 - f_3h_3 - f_1h_1)\,\mathrm{d}x_2 + f_3h_2\,\mathrm{d}x_3 + f_1h_2\,\mathrm{d}x_1 = 0\\
&(1 - f_1h_1 - f_2h_2)\,\mathrm{d}x_3 + f_1h_3\,\mathrm{d}x_1 + f_2h_3\,\mathrm{d}x_2 = 0.
\end{aligned} \tag{10}$$

Let D be the coefficient determinant of the $\mathrm{d}x_i$ in system (10). By a suitable choice of h_1, h_2 and h_3 we can obtain that D is zero. This is in fact equivalent to the equation

$$f_1h_1 + f_2h_2 + f_3h_3 = 1 \tag{11}$$

being satisfied. The fact that $D = 0$ implies that there indeed exists an M as above.

Substituting condition (11) into (10) we get

$$f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2 + f_3\,\mathrm{d}x_3 = 0. \tag{12}$$

From assumption (5) we know that the differential equation (12) is exactly solvable, in the sense that there exists a smooth (potential) function $F \in \mathcal{F}$ and a constant c such that

$$F(x) = c \tag{13}$$

and $\partial F/\partial x_i = f_i$. The above manifold M is then described by (13).

Therefore for any given integrable Poisson algebra $\mathcal{A}$ there always exists a two-dimensional manifold of the form (13) on which X'_{x_i} in (7) is a Hamiltonian vector field and the Poisson bracket of the algebra $\mathcal{A}$ is given by X'_{x_i} according to formula (3),

$$[x_i, x_j]_{\mathrm{PB}} = -X'_{x_i} x_j = \sum_{k=1}^{3}\epsilon_{ijk} f_k.$$

The two-dimensional manifold defined by (13) is unique (once c is given). Hence an integrable Poisson algebra is uniquely given by the two-dimensional manifold M described by $F(x) = c$. □

Before investigating the Poisson algebraic structures on general two-dimensional symplectic manifolds, we see that if M is a two-dimensional manifold embedded in $\mathbb{R}^3$ and ω is a symplectic form on M, then for $\alpha(x) \neq 0$, $\forall\, x \in \mathbb{R}^3$, $\alpha^{-1}\omega$ is also a symplectic form on M. For $f, g, h \in \mathcal{F}(M)$, if $[f, g]_{\mathrm{PB}} = h$ on the symplectic manifold (M, ω), then on the symplectic manifold $(M, \alpha^{-1}\omega)$, $\alpha(x) \neq 0$, $\forall\, x \in \mathbb{R}^3$, one has $[f, g]_{\mathrm{PB}} = \alpha h$. Therefore, we say that on a two-dimensional manifold embedded in $\mathbb{R}^3$, a Poisson algebra A is, by definition, equivalent to a Poisson algebra B if the Poisson bracket on A is the same as that on B, multiplied by some common non-zero factor $\alpha(x)$, $\forall\, x \in \mathbb{R}^3$.

Theorem 2. For a given smooth two-dimensional symplectic manifold M embedded in $\mathbb{R}^3$ of the form $F(x) = 0$, $F \in \mathcal{F}$, $x \in \mathbb{R}^3$, x generates a Poisson algebra with the following Poisson bracket:

$$[x_i, x_j]_{\mathrm{PB}} = \alpha(x)\sum_{k=1}^{3}\epsilon_{ijk}\frac{\partial F(x)}{\partial x_k} \tag{14}$$

$\alpha(x) \neq 0$, $\forall\, x \in \mathbb{R}^3$. This is unique in the sense of the above algebraic equivalence.

Proof. Let the symplectic form ω on M be given as

$$\omega = -\frac{1}{2}\sum_{i,j,k=1}^{3} \epsilon_{ijk} h'_i \, \mathrm{d}x_j \wedge \mathrm{d}x_k.$$

Let X_x be a vector field on M of the form

$$X_{x_i} = \sum_{j,k=1}^{3} \epsilon_{ijk} f'_j \frac{\partial}{\partial x_k} \qquad i = 1, 2, 3 \tag{15}$$

for some h'_i, $f'_i \in \mathcal{F}$, $i = 1, 2, 3$. In order for X_x to be the Hamiltonian vector field associated with ω we must have $X_{x_i} \rfloor \omega = -\mathrm{d}x_i$, thus we must have that

$$\begin{aligned}
&(1 - f'_2h'_2 - f'_3h'_3)\,\mathrm{d}x_1 + f'_2h'_1\,\mathrm{d}x_2 + f'_3h'_1\,\mathrm{d}x_3 = 0\\
&(1 - f'_3h'_3 - f'_1h'_1)\,\mathrm{d}x_2 + f'_3h'_2\,\mathrm{d}x_3 + f'_1h'_2\,\mathrm{d}x_1 = 0\\
&(1 - f'_1h'_1 - f'_2h'_2)\,\mathrm{d}x_3 + f'_1h'_3\,\mathrm{d}x_1 + f'_2h'_3\,\mathrm{d}x_2 = 0
\end{aligned} \tag{16}$$

where $\mathrm{d}x$ are not independent since $F(x) = 0$ implies

$$\sum_{i=1}^{3} \frac{\partial F(x)}{\partial x_i}\,\mathrm{d}x_i = 0. \tag{17}$$

Therefore, the coefficient determinant of the system (16) is zero, which gives

$$\sum_{i=1}^{3} f'_i h'_i = 1.$$

Hence the system of equations (16) becomes

$$\sum_{i=1}^{3} f'_i\,\mathrm{d}x_i = 0. \tag{18}$$

Equations (17) and (18) give rise to

$$f'_i(x) = \alpha(x)\frac{\partial F(x)}{\partial x_i} \qquad i = 1, 2, 3 \tag{19}$$

where $\alpha(x) \neq 0$, $\forall\, x \in \mathbb{R}^3$.

From (19) the Hamiltonian vector field (15) takes the form

$$X_{x_i} = \alpha(x)\sum_{j,k=1}^{3} \epsilon_{ijk}\frac{\partial F(x)}{\partial x_j}\frac{\partial}{\partial x_k}. \tag{20}$$

Using formula (3) we have

$$[x_i, x_j]_{\mathrm{PB}} = \alpha(x)\sum_{k=1}^{3} \epsilon_{ijk}\frac{\partial F(x)}{\partial x_k}. \tag{21}$$

This is just formula (14). □

Theorem 2 is more complete and general than the conclusion in [10]. When it is applied to such manifolds as the undeformed (respectively, q-deformed) two-dimensional sphere, the one sheet hyperboloid and the elliptic paraboloid, one gets [10] the Lie (respectively, q-deformed) algebra of $SU(2)$, $SU(1,1)$ and the harmonic oscillator algebra $\mathcal{H}(4)$ [11]. Here one notes that if $F(x)=0$ defines a smooth two-dimensional symplectic manifold M in $\mathbb{R}^3$, then $\alpha(x)F(x)=0$, $\alpha(x)\neq 0$, $\forall\, x\in\mathbb{R}^3$, also defines the same manifold M. By formula (14) we see that $F(x)=0$ and $\alpha(x)F(x)=0$ give rise to the same Poisson algebra under the algebraic equivalence we stated before theorem 2.

As $F\in\mathcal{F}$, we have that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial F(x)}{\partial x_i}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial F(x)}{\partial x_j}\right)\qquad i,j=1,2,3.$$

Therefore, the Poisson algebra given by (14) is by definition integrable and it is uniquely given by the manifold M. It is also direct to check that $F(x)$ is the centre of the Poisson algebra, i.e. $[x_i,F(x)]_{\rm PB}=0$, $i=1,2,3$. Moreover, from the Poisson algebraic relations (14) one has

$$[f,g]_{\rm PB}(x)=-\sum_{i,j,k=1}^{3}\epsilon_{ijk}\frac{\partial f}{\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial g}{\partial x_k}(x) \tag{22}$$

for $f,g\in\mathcal{F}$.

Theorems 1 and 2 establish the correspondence between two-dimensional symplectic manifolds and Poisson algebras with three generators. In what follows we study some properties related to smooth Poisson algebraic maps.

Let A (respectively B) be two integrable Poisson algebras with related two-dimensional manifolds M_A (respectively M_B) defined by $F_A(x)=0$ (respectively $F_B(y)=0$) in $\mathbb{R}^3$, where $x=(x_1,x_2,x_3)$ (respectively $y=(y_1,y_2,y_3)$) are the generators of the algebra A (respectively B). We call a smooth map $\tilde{y}(x)$, x as before and $\tilde{y}$ a generator of the Poisson algebra B, a smooth Poisson algebraic map.

Theorem 3. If the smooth Poisson algebraic map $\tilde{y}(x)=(\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)(x)$ between integrable Poisson algebras A and B satisfies the commutator relations of B, then $\tilde{y}$ satisfies $F_B(\tilde{y})=0$.

Proof. From theorem 2 the Poisson algebra A is given by

$$[x_i,x_j]_{\rm PB}=\sum_{i,j,k=1}^{3}\epsilon_{ijk}\frac{\partial F_A(x)}{\partial x_k}.$$

Using formula (22) we have

$$[\tilde{y}_i,\tilde{y}_j]_{\rm PB}(x)=-\sum_{l,m,n=1}^{3}\epsilon_{lmn}\frac{\partial \tilde{y}_i}{\partial x_l}\frac{\partial F_A}{\partial x_m}\frac{\partial \tilde{y}_j}{\partial x_n}(x). \tag{23}$$

Since $F_A(x)=0$, we have that the x_i, $i=1,2,3$, are not independent. Without loss of generality, we take x_1 and x_2 to be independent. By using the relation

$$\sum_{i=1}^{3}\frac{\partial F_A(x)}{\partial x_i}\,{\rm d}x_i=0$$

equation (23) becomes

$$[\tilde{y}_i,\tilde{y}_j]_{\rm PB}(x)=\frac{\partial F_A}{\partial x_3}\left(\frac{\partial \tilde{y}_i}{\partial x_1}\frac{\partial \tilde{y}_j}{\partial x_2}-\frac{\partial \tilde{y}_i}{\partial x_2}\frac{\partial \tilde{y}_j}{\partial x_1}\right)(x). \tag{24}$$

From theorem 2 the Poisson algebra B is given by

$$[y_i, y_j]_{\mathrm{PB}} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(y)}{\partial y_k}.$$

Hence if the smooth map $\tilde{y}(x) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)(x)$ satisfies the commutator relation of the algebra B, then

$$[\tilde{y}_i, \tilde{y}_j]_{\mathrm{PB}}(x) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(\tilde{y})}{\partial \tilde{y}_k}(x). \tag{25}$$

From (24) and (25) we obtain

$$\frac{\partial F_A}{\partial x_3} \left(\frac{\partial \tilde{y}_i}{\partial x_1} \frac{\partial \tilde{y}_j}{\partial x_2} - \frac{\partial \tilde{y}_i}{\partial x_2} \frac{\partial \tilde{y}_j}{\partial x_1} \right) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(\tilde{y})}{\partial \tilde{y}_k}. \tag{26}$$

Equation (26) represents three different equations for $i = 1,\ j = 2$; $i = 2,\ j = 3$; and $i = 3,\ j = 1$. Multiplying these equations by $\partial \tilde{y}_3 / \partial x_l$, $\partial \tilde{y}_1 / \partial x_l$ and $\partial \tilde{y}_2 / \partial x_l$, $l = 1, 2$, respectively, and summing these equations together we get

$$\frac{\partial F_B(\tilde{y})}{\partial x_l} = 0 \qquad l = 1, 2.$$

Therefore $F_B(\tilde{y})$ is independent of x_l, $l = 1, 2$, and thus $F_B(\tilde{y}) = \text{constant}$. This constant can be taken to be zero, since addition of a constant does not change the Poisson algebraic structure of the manifold. □

Theorem 4. If the map $\tilde{y}(x) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)(x)$ satisfies $F_B(\tilde{y}) = 0$, where x satisfies $F_A(x) = 0$, then $\tilde{y}$ generates the Poisson algebra B.

Proof. From theorem 2 we know that there is a unique Poisson algebra B associated with the manifold M_B (up to the algebraic equivalence). Hence if $\tilde{y}$ satisfies $F_B(\tilde{y}) = 0$, then $\tilde{y}$ generates the algebra B. □

Summarizing, we have discussed the relations between integrable Poisson algebraic structures and two-dimensional symplectic manifolds and have proved that there is a unique relation between integrable Poisson algebras and two-dimensional symplectic manifolds. We have also shown that a sufficient and necessary condition for a smooth Poisson algebraic map $\tilde{y}(x)$ to act from an integrable Poisson algebra A into an integrable Poisson algebra B is that both $F_A(x) = 0$ and $F_B(\tilde{y}) = 0$ are satisfied. The latter conclusions can be extended to the infinite-dimensional case, see [12].

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